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2007 J. Phys. A: Math. Theor. 40 13607

(http://iopscience.iop.org/1751-8121/40/45/008)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 13607–13623 doi:10.1088/1751-8113/40/45/008

An N-soliton solution to the DNLS equation based on revised inverse scattering transform

Guo-Quan Zhou and Nian-Ning Huang

Department of Physics, Wuhan University, Wuhan, 430072, People's Republic of China

E-mail: zgq@whu.edu.cn

Received 19 April 2007, in final form 15 August 2007
Published 23 October 2007
Online at stacks.iop.org/JPhysA/40/13607

Abstract

Based on a revised version of inverse scattering transform for the derivative nonlinear Schrödinger (DNLS) equation with vanishing boundary condition (VBC), the explicit N-soliton solution has been derived by some algebra techniques of some special matrices and determinants, especially the Binet–Cauchy formula. The one- and two-soliton solutions have been given as the illustration of the general formula of the N-soliton solution. Moreover, the asymptotic behaviors of the N-soliton solution have been discussed.

PACS numbers: 05.45.Yv, 42.81.Dp, 52.35.Sb

1. Introduction

As is well known, the derivative nonlinear Schrödinger (DNLS) equation is one of the rare several integrable nonlinear models that permit soliton solutions. It has been found that many physical phenomena, such as Alfvén waves in space plasma [1–8], sub-picosecond or femtosecond pulses in single-mode optical fibers [9–13], the weak nonlinear electromagnetic waves in (anti-)ferromagnetic or dielectric systems [14–16] under external magnetic fields, can all be described with the DNLS equation. Research of the DNLS equation has not only mathematic interest and significance, but also an important physical application background. The DNLS equation with different boundary conditions—the vanishing boundary condition (VBC) and the nonvanishing boundary condition (NVBC)—has been studied for a long time.

For the NVBC case, many heuristic and useful results have been attained [17–20]. Recently, by appropriately introducing an affine parameter in the Zakharov–Shabat (ZS) integral kern [21], Chen [19] and others have found its N-soliton solution for a special case that all the simple poles (zeros of $a(\lambda)$) locate on a circle of radius ρ centered at the origin. Lashkin [20] has found his multi-soliton solution for some extended case with N poles on a circle and M poles out of the circle. He has further developed its perturbation theory based on inverse scattering transform (IST).

For the VBC case, which is the only concerned theme of the present paper, some attempts and progresses have been made to solve the DNLS equation. Since Kaup and Newell [22] proposed an IST with a revision in their pioneer works, one-soliton solution was firstly attained and several versions of raw or explicit multi-soliton solutions were also obtained by means of different approaches [18, 23, 24]. Huang and Chen have obtained an N-soliton solution by means of Daboux transformation [24]. Steudel [18] has derived a formula for the N-soliton solution in terms of Vandermonde-like determinants by means of Bäcklund transformation [18]; but as [19] points out, his multi-soliton solution is difficult to demonstrate collisions among solitons and still has a too complex form to be used in the soliton perturbation theory of DNLSE, although it can easily generate compute pictures.

Since the integral kern in the Zakharov–Shabat (ZS) equation does not tend to zero in the limit of spectral parameter λ with $|\lambda| \to \infty$, the contribution of the path integral along the big circle (the out contour) is also nonvanishing; the usual procedure to perform inverse scattering transform encounters difficulty and is obsolete. Kaup thus proposed a revised IST by multiplying an additional weighing factor before the Jost solution $E(x, \lambda)$ so that it tends to zero as $|\lambda| \to \infty$; thus, the modified ZS kern should lead to a vanishing contribution of the integral along the big circle of the Cauchy contour. Though the one-soliton solution has been found by the obtained ZS equation of their IST, it is very difficult to derive directly its multi-soliton solution by their IST due to the existence of a complication phase factor which is related to the solution itself [22]. We thus consider proposing a new revised IST to avoid the excessive complexity. Our N-soliton solution obviously has a standard multi-soliton form. It can easily be used to discuss its asymptotic behaviors and then develop its direct perturbation theory. On the other hand, in solving the ZS equation for DNLS with VBC, we will unavoidably encounter a problem of calculating determinant $\det(I + Q_1 Q_2)$, for two $N \times N$ matrices Q_1 and Q_2, where I is an $N \times N$ identity matrix. Our paper also shows that the Binet–Cauchy formula and some other linear algebra techniques (appendices A.1–4) play important roles in the whole course, and are actually also effective for some other nonlinear integrable models [25].

This paper is organized as follows. In section 2, we review the general theory of IST for the DNLS equation with VBC, give the newly revised IST and the ZS equation by introducing a suitable factor λ^{-2} or λ^{-1} in the usual ZS integral kern. In section 3, the new ZS equation is solved and the raw expression of the N-soliton solution is expressed in a standard form. In section 4, the verification of the standard form for the N-soliton solution is provided. The explicit and unified N-soliton solution to the DNLS equation with VBC in the reflectionless case has been derived. In section 5, one- and two-soliton solutions have been given as two illustrations of the general formula for the N-soliton solution and the general computation procedures. In section 6, we have discussed in detail the asymptotic behaviors of our N-soliton solution. In section 7, we transform the N-soliton solution of the DNLS equation into that of the MNLS equation by gauge-like transformation. Some concluding remarks have been made in section 8. In the end, we have enumerated the needed algebra techniques in the appendices.

2. The revised inverse scattering transform and the Zakharov–Shabat equation for the DNLS equation with VBC

2.1. The fundamental concepts for the IST theory of the DNLS equation

The DNLS equation is usually expressed as

$$iu_t + u_{xx} + i(|u|^2 u)_x = 0 \qquad (1)$$

with vanishing boundary condition, where the subscripts stand for partial derivative. Its Lax pair is given by

$$L = -\mathrm{i}\lambda^2\sigma_3 + \lambda U, \qquad U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \tag{2}$$

and

$$M = -\mathrm{i}2\lambda^4\sigma_3 + 2\lambda^3 U - \mathrm{i}\lambda^2 U^2\sigma_3 - \lambda(-U^3 + \mathrm{i}U_x\sigma_3) \tag{3}$$

where λ is a spectral parameter, and σ_3 is the third one of Pauli matrices σ_1,σ_2,σ_3, and a letter with a bar (e.g. $\bar{u}$ in (2)) represents complex conjugate. The first Lax equation is

$$\partial_x f(x,\lambda) = L(x,\lambda) f(x,\lambda). \tag{4}$$

In the limit of $|x| \to \infty$, $u \to 0$, and

$$L \to L_0 = -\mathrm{i}\lambda^2\sigma_3 \qquad M \to M_0 = -\mathrm{i}2\lambda^4\sigma_3. \tag{5}$$

The free Jost solution is a 2×2 matrix

$$E(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda^2 x\sigma_3}, \qquad E_1(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\lambda^2 x} \qquad E_2(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\lambda^2 x}. \tag{6}$$

The Jost solutions of (4) are defined by their asymptotic behaviors as $x \to \pm\infty$,

$$\Psi(x,\lambda) = (\tilde{\psi}(x,\lambda),\ \psi(x,\lambda)) \to E(x,\lambda), \qquad \text{as} \quad x \to \infty \tag{7}$$

$$\Phi(x,\lambda) = (\phi(x,\lambda), \tilde{\phi}(x,\lambda)) \to E(x,\lambda), \qquad \text{as} \quad x \to -\infty \tag{8}$$

where $\psi(x,\lambda) = (\psi_1(x,\lambda), \psi_2(x,\lambda))^T$, $\tilde{\psi}(x,\lambda) = (\tilde{\psi}_1(x,\lambda), \tilde{\psi}_2(x,\lambda))^T$, etc, and superscript '$T$' represents transposing of a matrix here and afterward.

Since the first Lax equation of DNLS is similar to that of NLS, there are some similar properties of the Jost solutions. The monodromy matrix $T(\lambda)$ is defined as

$$\Phi_{(x,\lambda)} = \Psi(x,\lambda) T(\lambda) \tag{9}$$

where

$$T(\lambda) = \begin{pmatrix} a(t,\lambda) & -\tilde{b}(t,\lambda) \\ b(t,\lambda) & \tilde{a}(t,\lambda) \end{pmatrix}. \tag{10}$$

It is easy to find from (2) and (9) that

$$\sigma_2\, \overline{L(\bar{\lambda})}\, \sigma_2 = L(\lambda), \qquad \sigma_2\, \overline{T(\bar{\lambda})}\, \sigma_2 = T(\lambda) \tag{11}$$

$$\sigma_2\, \overline{\Psi(x,\bar{\lambda})}\, \sigma_2 = \Psi(x,\lambda), \qquad \sigma_2\, \overline{\Phi(x,\bar{\lambda})}\, \sigma_2 = \Phi(x,\lambda) \tag{12}$$

and

$$\sigma_3\Psi(x,\lambda)\sigma_3 = \Psi(x,-\lambda), \qquad \sigma_3\Phi(x,\lambda)\sigma_3 = \Phi(x,-\lambda) \tag{13}$$

$$\sigma_3 L(\lambda)\sigma_3 = L(-\lambda), \qquad \sigma_3 T(\lambda)\sigma_3 = T(-\lambda). \tag{14}$$

Then we can get following reduction relation and symmetry properties:

$$\mathrm{i}\sigma_2\overline{\psi(x,\bar{\lambda})} = \tilde{\psi}(x,\lambda) \tag{15}$$

$$-\mathrm{i}\sigma_2\bar{\phi}(x,\bar{\lambda}) = \tilde{\phi}(x,\lambda) \tag{16}$$

$$\bar{\tilde{a}}(\bar{\lambda}) = a(\lambda), \qquad \bar{\tilde{b}}(\bar{\lambda}) = b(\lambda) \tag{17}$$

and

$$\psi(x,-\lambda) = -\sigma_3\psi(x,\lambda) \tag{18}$$

$$\tilde{\psi}(x,-\lambda) = \sigma_3\tilde{\psi}(x,\lambda) \tag{19}$$

$$\begin{aligned} a(-\lambda) &= a(\lambda), \qquad & b(-\lambda) &= -b(\lambda) \\ \tilde{a}(-\lambda) &= \tilde{a}(\lambda), \qquad & \tilde{b}(-\lambda) &= -\tilde{b}(\lambda). \end{aligned} \tag{20}$$

2.2. Relation between the Jost functions and solutions to the DNLS equation

The asymptotic behaviors of the Jost solutions in the limit of $|\lambda| \to \infty$ can be obtained by simple derivation. Let $\upsilon = (\upsilon_1, \upsilon_2)^T \equiv \tilde{\psi}(x, \lambda)$, and equation (4) can be rewritten as

$$\upsilon_{1x} + i\lambda^2\upsilon_1 = \lambda u\upsilon_2, \qquad \upsilon_{2x} - i\lambda^2\upsilon_2 = -\lambda\bar{u}\upsilon_1. \tag{21}$$

Then we have

$$\upsilon_{1xx} - \frac{u_x}{u}(\upsilon_{1x} + i\lambda^2\upsilon_1) + \lambda^4\upsilon_1 + \lambda^2|u|^2\upsilon_1 = 0. \tag{22}$$

In the limit $|\lambda| \to \infty$, we assume

$$\tilde{\psi}_1(x, \lambda) = e^{-i\lambda^2 x + g}.$$

Substituting it into equation (22), we have

$$(-i\lambda^2 + g_x)^2 + g_{xx} - \frac{u_x}{u}g_x + \lambda^4 + \lambda^2|u|^2 = 0. \tag{23}$$

In the limit $|\lambda| \to \infty$, g_x can be expanded as series of $(\lambda^{-2})^j$, $j = 1, 2, \ldots$

$$ig_x \equiv \mu = \mu_0 + \mu_2(2\lambda^2)^{-1} + \cdots \tag{24}$$

and

$$\mu_0 = \tfrac{1}{2}|u|^2, \qquad \mu_2 = -i\tfrac{1}{2}\bar{u}_x u - \tfrac{1}{4}|u|^4, \ldots. \tag{25}$$

Equation (21) leads to $g_x\upsilon_1 = \lambda u\upsilon_2$. Considering (25), in the limit of $|\lambda| \to \infty$, we find a useful formula

$$\bar{u} = i2 \lim_{|\lambda|\to\infty} \frac{\lambda\tilde{\psi}_2(x, \lambda)}{\tilde{\psi}_1(x, \lambda)} \tag{26}$$

which expresses the solution u in terms of the Jost solutions as $|\lambda| \to \infty$.

On the other hand, the zeros of $a(\lambda)$ appear in pairs, and can be designed by λ_n, $n = 1, 2, \ldots, N$ in the I quadrant, and $\lambda_{n+N} = -\lambda_n$ in the III quadrant. The discrete part of $a(\lambda)$ is [26, 27]

$$a(\lambda) = \prod_{n=1}^{N} \frac{\lambda^2 - \lambda_n^2}{\lambda^2 - \overline{\lambda_n^2}} \frac{\overline{\lambda_n^2}}{\lambda_n^2} \tag{27}$$

where $a(0) = 1$. It comes from our consideration of the fact that, from the sum of two Cauchy integrals

$$\frac{\ln a(\lambda)}{\lambda} + 0 = \frac{1}{2\pi i}\int_\Gamma d\lambda' \frac{\ln a(\lambda')\tilde{a}(\lambda')}{\lambda'(\lambda' - \lambda)}, \qquad \Gamma = (0, \infty) \cup (i\infty, i0) \cup (0, -\infty) \cup (-i\infty, i0),$$

in order to maintain that

$$\ln a(\lambda) \to 0, \quad \text{as} \quad \lambda \to 0, \quad \text{and} \quad \ln a(\lambda) \text{ is finite as } |\lambda| \to \infty,$$

we then have to introduce a factor $\overline{\lambda_n^2}/\lambda_n^2$ in (27).

At the zeros of $a(\lambda)$, we have

$$\phi(x, \lambda_n) = b_n\psi(x, \lambda_n), \qquad \dot{a}(-\lambda_n) = -\dot{a}(\lambda_n), \qquad b_{n+N} = -b_n. \tag{28}$$

As $\mu_0 \neq 0$, the Jost solutions do not tend to the free Jost solutions $E(x, \lambda)$ in the limit of $|\lambda| \to \infty$. This is their most typical property, which means that the usual procedure of constructing the equation of IST by a Cauchy contour integral must be invalid; thus, a newly revised procedure to derive a suitable IST and the corresponding ZS equation is proposed in our group.

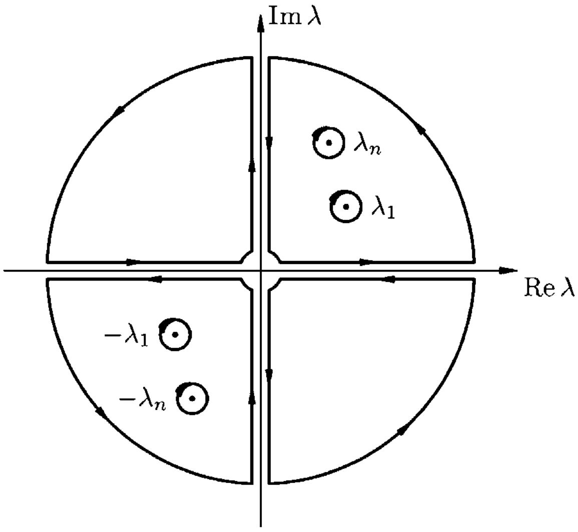

Figure 1. The integral path for IST of the DNLS.

2.3. The revised IST and Zakharov–Shabat equation for DNLSE with VBC

The 2×1 column function $\Theta(x, \lambda)$ can be introduced as usual

$$\Theta(x,\lambda) = \begin{cases} \dfrac{1}{a(\lambda)}\phi(x,\lambda), & \text{as } \lambda \text{ in I, III quadrants} \\ \tilde{\psi}(x,\lambda), & \text{as } \lambda \text{ in II, IV quadrants.} \end{cases} \tag{29}$$

An alternative form of the IST equation is proposed as

$$\frac{1}{\lambda^2}\{\Theta_1(x,\lambda) - E_{11}(x,\lambda)\}\,\mathrm{e}^{\mathrm{i}\lambda^2 x} = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathrm{d}\lambda' \frac{1}{\lambda'-\lambda}\frac{1}{\lambda'^2}\{\Theta_1(x,\lambda') - E_{11}(x,\lambda')\}\,\mathrm{e}^{\mathrm{i}\lambda'^2 x}. \tag{30}$$

Because in the limit of $|\lambda| \to \infty$

$$\lim_{|\lambda|\to\infty} \mathrm{e}^{\mathrm{i}\lambda^2 x} = 0, \qquad \text{as} \begin{cases} x > 0, & \mathrm{Im}\,\lambda^2 > 0, \quad (\lambda \text{ in the I, III quadrants}), \\ x < 0, & \mathrm{Im}\,\lambda^2 < 0, \quad (\lambda \text{ in the II, IV quadrants}), \end{cases}$$

so the integral path Γ should be selected as shown in figure 1, where the radius of big circle tends to infinite, while the radius of small circle tends to zero. The factor λ^{-2} is introduced to ensure that the contribution of the integral along the big arc is vanishing. Meanwhile, our modification produces no new poles since the Lax operator $L \to 0$, as $\lambda \to 0$.

In the reflectionless case, the revised IST equation gives

$$\tilde{\psi}_1(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda^2 x} + \sum_{n=1}^{2N} \frac{1}{\lambda_n^2}\frac{\lambda^2}{\lambda-\lambda_n}\frac{b_n}{\dot{a}(\lambda_n)}\psi_1(x,\lambda_n)\,\mathrm{e}^{\mathrm{i}\lambda_n^2 x}\,\mathrm{e}^{-\mathrm{i}\lambda^2 x} \tag{31}$$

where $\dot{a}(\lambda_n) = \frac{\mathrm{d}}{\mathrm{d}\lambda}a(\lambda)\big|_{\lambda=\lambda_n}$. Similarly, an alternative form of IST equation is proposed as

$$\frac{1}{\lambda}\{\Theta_2(x,\lambda)\}\,\mathrm{e}^{\mathrm{i}\lambda^2 x} = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathrm{d}\lambda' \frac{1}{\lambda'-\lambda}\frac{1}{\lambda'}\{\Theta_2(x,\lambda')\}\,\mathrm{e}^{\mathrm{i}\lambda'^2 x} \tag{32}$$

where a factor λ^{-1} is introduced for the same reason as λ^{-2} in equation (30). Then in the reflectionless case, we can attain

$$\tilde{\psi}_2(x,\lambda) = \sum_{n=1}^{2N} \frac{1}{\lambda_n}\frac{\lambda}{\lambda-\lambda_n}\frac{b_n}{\dot{a}(\lambda_n)}\psi_2(x,\lambda_n)\,\mathrm{e}^{\mathrm{i}\lambda_n^2 x}\,\mathrm{e}^{-\mathrm{i}\lambda^2 x}. \tag{33}$$

Taking the symmetry and reduction relation (18) and (28) into consideration, from (31) and (33) we can obtain the revised Zakharov–Shabat equation for the DNLS equation with VBC, that is

$$\tilde{\psi}_1(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda^2 x} + \sum_{n=1}^{N} \frac{2\lambda^2}{\lambda_n\left(\lambda^2-\lambda_n^2\right)} \frac{b_n}{\dot{a}(\lambda_n)} \psi_1(x_1,\lambda_n)\,\mathrm{e}^{\mathrm{i}\lambda_n^2 x}\,\mathrm{e}^{-\mathrm{i}\lambda^2 x} \tag{34}$$

$$\tilde{\psi}_2(x,\lambda) = \sum_{n=1}^{N} \frac{2\lambda}{\lambda^2-\lambda_n^2} \frac{b_n}{\dot{a}(\lambda_n)} \psi_2(x_s,\lambda_n)\,\mathrm{e}^{\mathrm{i}\lambda_n^2 x}\,\mathrm{e}^{-i\lambda^2 x}. \tag{35}$$

3. The raw expression of the *N*-soliton solution

Substituting equations (34) and (35) into formula (26), we thus attain the N-soliton solution

$$\bar{u}_N = -\mathrm{i}2\frac{U_N}{V_N} \tag{36}$$

where

$$\begin{aligned} U_N &= \sum_{n=1}^{N} \frac{2b_n}{\dot{a}(\lambda_n)} \psi_2(x,\lambda_n)\,\mathrm{e}^{\mathrm{i}\lambda_n^2 x} \\ V_N &= 1 + \sum_{n=1}^{N} \frac{2b_n}{\lambda_n \dot{a}(\lambda_n)} \psi_1(x,\lambda_n)\,\mathrm{e}^{\mathrm{i}\lambda_n^2 x}. \end{aligned} \tag{37}$$

Let $\lambda = \bar{\lambda}_m$, $m = 1, 2, \ldots, N$, respectively, in equations (34) and (35), and making use of the symmetry and reduction relation (15), we can attain

$$\overline{\psi_2}(x,\lambda_m) = \mathrm{e}^{-\mathrm{i}\overline{\lambda_m^2}x} + \sum_{n=1}^{N} \frac{2\overline{\lambda_m^2}}{\lambda_n\left(\overline{\lambda_m^2}-\lambda_n^2\right)} c_n \psi_1(x,\lambda_n)\,\mathrm{e}^{\mathrm{i}\lambda_n^2 x}\,\mathrm{e}^{-\mathrm{i}\overline{\lambda_m^2}x} \tag{38}$$

$$\overline{\psi}_1(x,\lambda_m) = -\sum_{n=1}^{N} \frac{2\overline{\lambda_m^2}}{\overline{\lambda_m^2}-\lambda_n^2} c_n \psi_2(x,\lambda_n)\,\mathrm{e}^{\mathrm{i}\lambda_n^2 x}\,\mathrm{e}^{-\mathrm{i}\overline{\lambda_m^2}x} \qquad m = 1, 2, \ldots, N, \tag{39}$$

where $c_n = b_n/\dot{a}(\lambda_n)$. We also define

$$f_n = \sqrt{2c_n}\,\mathrm{e}^{\mathrm{i}\lambda_n^2 x}, \qquad (w_j)_n = \sqrt{2c_n}\psi_j(\lambda_n) \qquad j = 1, 2, \qquad \text{and} \qquad n = 1, 2, \ldots, N. \tag{40}$$

$$(B_1)_{mn} = \overline{f_m} \frac{\overline{\lambda_m^2}}{\left(\overline{\lambda_m^2}-\lambda_n^2\right)\lambda_n} f_n, \qquad (B_2)_{mn} = \bar{f}_m \frac{\overline{\lambda_m}}{\bar{\lambda}_m^2-\lambda_n^2} f_n, \qquad m, n = 1, 2, \ldots, N \tag{41}$$

$$\begin{aligned} &W_1 = ((w_1)_1, (w_1)_2, \ldots, (w_1)_N)^T, \qquad W_2 = ((w_2)_1, (w_2)_2, \ldots, (w_2)_N)^T \\ &F = (f_1, f_2, \ldots, f_N)^T, \qquad G = \left(\frac{f_1}{\lambda_1}, \frac{f_2}{\lambda_2}, \ldots, \frac{f_N}{\lambda_N}\right)^T \end{aligned} \tag{42}$$

where superscript 'T' represents transposition of a matrix.

Then equations (38) and (39) can be rewritten as

$$(\overline{w_2})_m = \bar{f}_m + \sum_{n=1}^{N} (B_1)_{mn}(w_1)_n \tag{43}$$

$$(\overline{w_1})_m = -\sum_{n=1}^{N} (B_2)_{mn}(w_2)_n \tag{44}$$

where $m = 1, 2, \ldots, N$. They can be rewritten in a more compact matrix form:

$$\overline{W}_2 = \bar{F} + B_1 \bullet W_1 \tag{43a}$$

$$\overline{W}_1 = -B_2 \bullet W_2. \tag{44b}$$

Then

$$W_2 = (I + \bar{B}_1 B_2)^{-1} F \tag{45}$$

$$W_1 = -\bar{B}_2(I + B_1\bar{B}_2)^{-1}\bar{F} \tag{46}$$

where I is the $N \times N$ identity matrix. On the other hand, from (37) we know

$$U_N = \sum_{n=1}^{N} f_n w_{2n} = F^T W_2 \tag{47}$$

$$V_N = 1 + \sum_{n=1}^{N} \frac{f_n}{\lambda_n} w_{1n} = 1 + G^T W_1. \tag{48}$$

Substituting equations (45) and (46) into (47) and (48) and then substituting (47) and (48) into formula (36), we obtain

$$\begin{aligned}\bar{u}_N &= -\mathrm{i}2\frac{F^T W_2}{1 + G^T W_1} = -\mathrm{i}2\frac{F^T(I + \bar{B}_1 B_2)^{-1}F}{1 - G^T\bar{B}_2(I + B_1\bar{B}_2)^{-1}\bar{F}} \\ &= -\mathrm{i}2\frac{\det(I + \bar{B}_1 B_2 + FF^T) - \det(I + \bar{B}_1 B_2)}{\det[I + (B_1 - \bar{F}G^T)\bar{B}_2]} \times \frac{\det(I + B_1\bar{B}_2)}{\det(I + \bar{B}_1 B_2)} \equiv -2\mathrm{i}\frac{AD}{\bar{D}^2}\end{aligned} \tag{49}$$

where

$$A \equiv \det(I + \bar{B}_1 B_2 + FF^T) - \det(I + \bar{B}_1 B_2) \tag{50}$$

$$D \equiv \det(I + B_1\bar{B}_2). \tag{51}$$

In the subsequent section, we will prove that

$$\det[I + (B_1 - \bar{F}G^T)\bar{B}_2] = \det(I + \bar{B}_1 B_2) \equiv \bar{D}. \tag{52}$$

It is obvious that formula (49) has the usual standard form of a soliton solution.

Here in formula (49), some algebra techniques have been used and can be found in equation (A.1)

4. The explicit expression of the *N*-soliton solution

4.1. Verification of standard form for the N-soliton solution

We only need to prove that equation (52) holds.

Firstly, we define $N \times N$ matrices P_1, P_2, Q_1, Q_2 respectively as

$$(P_1)_{nm} \equiv (B_1 - \bar{F}G^T)_{nm} = \bar{f}_n \frac{\lambda_m}{\overline{\lambda_n^2} - \lambda_m^2} f_m$$
$$(P_2)_{mn} \equiv (\bar{B}_2)_{mn} = f_m \frac{\lambda_m}{\lambda_m^2 - \overline{\lambda_n^2}} \bar{f}_n \tag{53}$$

$$(Q_1)_{nm} \equiv (\bar{B}_1)_{nm} = f_n \frac{\lambda_n^2}{\lambda_n^2 - \overline{\lambda_m^2}} \left(\frac{\bar{f}_m}{\bar{\lambda}_m}\right)$$
$$(Q_2)_{mn} \equiv (B_2)_{mn} = \bar{f}_m \frac{\bar{\lambda}_m}{\overline{\lambda_m^2} - \lambda_n^2} f_n. \tag{54}$$

Then

$$\begin{aligned}\bar{D} &= \det(I + Q_1Q_2)\\ &= 1 + \sum_{r=1}^{N} \sum_{1\leqslant n_1<n_2<\cdots<n_r\leqslant N} \bar{D}_r(n_1, n_2, \ldots, n_r)\\ &= 1 + \sum_{r=1}^{N} \sum_{1\leqslant n_1<\cdots<n_r\leqslant N} \sum_{1\leqslant m_1<\cdots<m_r\leqslant N} Q_1(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)\\ &\quad \times Q_2(m_1, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r)\end{aligned} \tag{55}$$

where $Q_1(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)$ denotes a minor, which is the determinant of a submatrix of Q_1 consisting of elements belonging not only to rows $(n_1, n_2, \ldots, n_r)$ but also columns $(m_1, m_2, \ldots, m_r)$. Here the Binet–Cauchy formula is used in equations (A.2)–(A.4). Then

$$\begin{aligned}&Q_1(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)Q_2(m_1, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r)\\ &\quad = \prod_{n,m} \frac{f_n\bar{f}_m}{\lambda_n^2 - \bar{\lambda}_m^2} \frac{\lambda_n^2}{\bar{\lambda}_m} \prod_{n<n',m<m'} \left(\lambda_n^2 - \lambda_{n'}^2\right)\left(\lambda_{m'}^2 - \lambda_m^2\right) \bullet \prod_{m,n} \frac{\bar{f}_m f_n}{\bar{\lambda}_m^2 - \lambda_n^2} \bar{\lambda}_m\\ &\qquad \times \prod_{n<n',m<m'} \left(\bar{\lambda}_m^2 - \lambda_{m'}^2\right)\left(\lambda_{n'}^2 - \lambda_n^2\right)\\ &\quad = (-1)^r \prod_{m,n} \frac{\lambda_n^2 f_n^2 \overline{f_m^2}}{\left(\lambda_n^2 - \overline{\lambda_m^2}\right)^2} \prod_{n<n',m<m'} \left(\lambda_n^2 - \lambda_{n'}^2\right)^2 \left(\overline{\lambda_m^2} - \overline{\lambda_{m'}^2}\right)^2\end{aligned} \tag{56}$$

where

$$n, n' \in \{n_1, n_2, \ldots, n_r\}, \qquad m, m' \in \{m_1, m_2, \ldots, m_r\}. \tag{57}$$

Similarly,

$$\begin{aligned}&P_1(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)P_2(m_1, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r)\\ &\quad = (-1)^r \prod_{n,m} \frac{\bar{f}_n^2 f_m^2 \lambda_m^2}{\left(\bar{\lambda}_n^2 - \lambda_m^2\right)} \prod_{n<n',m<m'} \left(\lambda_m^2 - \lambda_{m'}^2\right)^2 \left(\overline{\lambda_n^2} - \overline{\lambda_{n'}^2}\right)^2\end{aligned} \tag{58}$$

where

$$n, n' \in \{n_1, n_2, \ldots, n_r\}, \qquad m, m' \in \{m_1, m_2, \ldots, m_r\} \tag{57a}$$

and

$$\det[I + (B_1 - \bar{F}G^T)\bar{B}_2] = \det(I + P_1P_2)$$
$$= 1 + \sum_{r=1}^{N} \sum_{1\leqslant n_1<\cdots<n_r\leqslant N} \sum_{1\leqslant m_1<\cdots<m_r\leqslant N} P_1(n_1, \ldots, n_r; m_1, \ldots, m_r)$$
$$\times P_2(m_1, \ldots, m_r; n_1, \ldots, n_r). \tag{59}$$

It is easy to find a kind of permutation symmetry existing between expressions (56) and (58), that is

$$P_1(n_1, \ldots, n_r; m_1, \ldots, m_r)P_2(m_1, \ldots, m_r; n_1, \ldots, n_r)$$
$$= Q_1(m_1, \ldots, m_r; n_1, \ldots, n_r)Q_2(n_1, \ldots, n_r; m_1, \ldots, m_r). \tag{60}$$

Comparing (55) with (59) and making use of (60), we complete our demonstration that equation (52) holds. Thus the soliton solution is surely of the standard form as usual and can be expressed as formula (49).

4.2. Introduction of time evolution function

The time evolution factor of the scattering data can be introduced by standard procedure [25]. Due to the fact that the second Lax operator $M \to -\mathrm{i}2\lambda^4\sigma_3$ in the limit $|x| \to \infty$, it is easy to derive the time dependence of scattering date

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\lambda_n = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}a(\lambda_n) = 0 \\ &c_n(t) = c_{n0}\,\mathrm{e}^{\mathrm{i}4\lambda_n^4 t}, \qquad c_{n0} = \frac{b_{n0}}{\dot{a}(\lambda_n)}, \qquad b_n(t) = b_{n0}\,\mathrm{e}^{\mathrm{i}4\lambda_n^4 t}. \end{aligned} \tag{61}$$

Then the typical soliton arguments θ_n and φ_n can be defined according to

$$f_n^2 = 2c_{n0}\,\mathrm{e}^{\mathrm{i}2\lambda_n^2 x}\mathrm{e}^{\mathrm{i}4\lambda_n^4 t} \equiv 2c_{n0}\,\mathrm{e}^{-\theta_n}\,\mathrm{e}^{\mathrm{i}\varphi_n} \tag{62}$$

where

$$\begin{aligned} &\theta_n = 4\mu_n\nu_n\big[x + 4\big(\mu_n^2 - \nu_n^2\big)t\big] = 4\kappa_n(x - V_n t) \\ &\varphi_n = 2\big(\mu_n^2 - \nu_n^2\big)x + \big[4\big(\mu_n^2 - \nu_n^2\big)^2 - 16\mu_n^2\nu_n^2\big] \bullet t \\ &\lambda \equiv \mu_n + \mathrm{i}\nu_n, \qquad V_n = -4\big(\mu_n^2 - \nu_n^2\big), \qquad \kappa_n = 4\mu_n\nu_n. \end{aligned} \tag{63}$$

4.3. Calculation of determinant of $\bar{D}$ and A

Substituting expressions (61) and (62) into formula (56) and then into (55), we have

$$Q_1(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)Q_2(m_1, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r)$$
$$= (-1)^r \prod_{n,m} (2c_n)(2\bar{c}_m)\,\mathrm{e}^{-\theta_n}\,\mathrm{e}^{\mathrm{i}\varphi_n}\,\mathrm{e}^{-\theta_m}\,\mathrm{e}^{-\mathrm{i}\varphi_m}\,\frac{\lambda_n^2}{\big(\lambda_n^2 - \overline{\lambda_m^2}\big)^2}$$
$$\times \prod_{n<n',m<m'} \big(\lambda_n^2 - \lambda_{n'}^2\big)^2\big(\overline{\lambda_m^2} - \overline{\lambda_{m'}^2}\big)^2 \tag{64}$$

with $n, n' \in \{n_1, n_2, \ldots, n_r\}$, and $m, m' \in \{m_1, m_2, \ldots, m_r\}$.

Where the Binet–Cauchy formula is applied, which is numerated in equations (A.3) and (A.4). Substituting expression (64) into formula (55) completes the calculation of determinant $\bar{D}$.

In the calculation of the most complicated determinant A in (49), we introduce an $N \times (N+1)$ matrix Ω_1 and an $(N+1) \times N$ matrix Ω_2 defined as

$$\begin{aligned} (\Omega_1)_{nm} &= (\bar{B}_1)_{nm} = (Q_1)_{nm}, \qquad (\Omega_1)_{n0} = f_n \\ (\Omega_2)_{mn} &= (B_2)_{mn} = (Q_2)_{mn}, \qquad (\Omega_2)_{0n} = f_n \end{aligned} \tag{65}$$

with $n, m = 1, 2, \ldots, N$.

We thus have

$$\begin{aligned} \det(I + \bar{B}_1 B_2 + F F^T) &= \det(I + \Omega_1 \Omega_2) \\ &= 1 + \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < \cdots < n_r \leqslant N} \sum_{0 \leqslant m_1 < \cdots < m_r \leqslant N} \Omega_1(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) \\ &\quad \times \Omega_2(m_1, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r). \end{aligned} \tag{66}$$

The above summation obviously can be decomposed into two parts: one is extended to $m_1 = 0$, the other extended to $m_1 \geqslant 1$. Subtracted from (66), the part that is extended to $m_1 \geqslant 1$, the remaining part of (66) is exactly A in (49) (with $m_1 = 0$ and $m_2 \geqslant 1$). Due to (65), we have

$$\begin{aligned} A &= \det(I + \Omega_1 \Omega_2) - \det(I + Q_1 Q_2) \\ &= \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < \cdots < n_r \leqslant N} \sum_{1 \leqslant m_2 < \cdots < m_r \leqslant N} A_r(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) \\ &= \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < n_2 < n_r \leqslant N}^{N} \sum_{1 \leqslant m_2 < m_3 < \cdots < m_r \leqslant N} \Omega_1(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) \\ &\quad \times \Omega_2(0, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r) \end{aligned}$$

with

$$\begin{aligned} &\Omega_1(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r)\Omega_2(0, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r) \\ &\quad = (-1)^{r-1} \prod_{n,m} \frac{f_n^2 \overline{f_m^2 \lambda_m^2}}{\left(\lambda_n^2 - \overline{\lambda_m^2}\right)^2} \prod_{n<n',m<m'} \left(\lambda_n^2 - \lambda_{n'}^2\right)^2 \left(\overline{\lambda_m^2} - \overline{\lambda_{m'}^2}\right)^2 \\ &\quad = (-1)^{r-1} \prod_{n,m} (2c_n)(2\overline{c_m})\, \mathrm{e}^{-\theta_n}\, \mathrm{e}^{\mathrm{i}\varphi_n}\, \mathrm{e}^{-\theta_m}\, \mathrm{e}^{-\mathrm{i}\varphi_m} \frac{\overline{\lambda_m^2}}{\left(\lambda_n^2 - \overline{\lambda_m^2}\right)^2} \\ &\quad\quad \times \prod_{n<n',m<m'} \left(\lambda_n^2 - \lambda_{n'}^2\right)^2 \left(\overline{\lambda_m^2} - \overline{\lambda_{m'}^2}\right)^2 \end{aligned} \tag{67}$$

where $n, n' \in \{n_1, n_2, \ldots, n_r\}$ and especially $m, m' \in \{m_2, \ldots, m_r\}$, which completes the calculation of determinant A in formula (49).

Substituting the explicit expressions of D, $\bar{D}$ and A into (49), we finally attain the explicit expression of the N-soliton solution to the DNLS equation in VBC and the reflectionless case, based upon a newly revised IST technique.

5. The typical examples for one- and two-soliton solutions

We give two concrete examples—the one- and two-soliton solutions as illustrations of the general explicit soliton solution.

In the case of one-soliton solution, $N = 1$, $\lambda_2 = -\lambda_1$, $\lambda_1 = \rho_1\,\mathrm{e}^{\mathrm{i}\beta_1} = \mu_1 + \mathrm{i}\nu_1$ and

$$A_1 = \Omega_1(n_1 = 1; m_1 = 0)\Omega_2(m_1 = 0; n_1 = 1) = f_1^2 \tag{68}$$

$$\bar{D}_1 = Q_1(n_1 = 1; m_1 = 1)Q_2(m_1 = 1; n_2 = 1) = 1 - |f_1|^4\lambda_1^2\big/\big(\lambda_1^2 - \overline{\lambda_1^2}\big)^2 \tag{69}$$

$$f_1^2 = 2c_{10}\,\mathrm{e}^{\mathrm{i}2\lambda^2 x}\,\mathrm{e}^{\mathrm{i}4\lambda_1^4 t} \qquad b_{10}\,\mathrm{e}^{\mathrm{i}2\lambda^2 x}\,\mathrm{e}^{\mathrm{i}4\lambda_1^4 t} \equiv \mathrm{e}^{-\theta_1}\,\mathrm{e}^{\mathrm{i}\varphi_1}$$

$$c_{10} = \frac{b_{10}\lambda_1\big(\lambda_1^2 - \overline{\lambda_1^2}\big)}{2\overline{\lambda_1^2}}, \qquad b_{10} = \mathrm{e}^{4\mu_1\nu_1 x_{10}}\,\mathrm{e}^{\mathrm{i}\alpha_{10}}.$$

It is different slightly from the definition in (63) for that here b_{10} has been absorbed into the soliton center and initial phase.

$$\begin{aligned} \theta_1 &= 4\mu_1\nu_1\big[x - x_{10} + 4\big(\mu_1^2 - \nu_1^2\big)t\big] \\ \varphi_1 &= 2\big(\mu_1^2 - \nu_1^2\big)x + \big[4\big(\mu_1^2 - \nu_1^2\big)^2 - 16\mu_1\nu_1^2\big]t + \alpha_{10} \end{aligned} \tag{70}$$

then

$$A_1 = \frac{\lambda_1\big(\lambda_1^2 - \overline{\lambda_1^2}\big)}{\overline{\lambda_1^2}}\,\mathrm{e}^{-\theta_1}\,\mathrm{e}^{\mathrm{i}\varphi_1} = \mathrm{i}2\rho_1\sin 2\beta_1\,\mathrm{e}^{\mathrm{i}3\beta_1}\,\mathrm{e}^{-\theta_1}\,\mathrm{e}^{\mathrm{i}\varphi_1}$$

$$\bar{D}_1 = 1 - \frac{\big|\lambda_1^2 - \overline{\lambda_1^2}\big|^2}{|\lambda_1|^2}\,\frac{\lambda_1^2}{\big(\lambda_1^2 - \overline{\lambda_1^2}\big)^2}\,\mathrm{e}^{-2\theta_1} = 1 + \mathrm{e}^{\mathrm{i}2\beta_1}\,\mathrm{e}^{-2\theta_1}$$

and

$$\bar{u}_1(x,t) = -\mathrm{i}2\cdot\frac{A_1 D_1}{\bar{D}_1^2} = \frac{4\rho_1\sin 2\beta_1\,\mathrm{e}^{\mathrm{i}3\beta_1}(1 + \mathrm{e}^{-\mathrm{i}2\beta_1}\,\mathrm{e}^{-2\theta_1})}{(1 + \mathrm{e}^{\mathrm{i}2\beta_1}\mathrm{e}^{-2\theta_1})^2}\cdot\mathrm{e}^{-\theta_1}\,\mathrm{e}^{\mathrm{i}\varphi_1}. \tag{71}$$

The complex conjugate of the one-soliton solution $\bar{u}$ in (71) is only in conformity with that obtained from pure Marchenko formalism [26], up to a permitted global constant phase factor.

In the case of the two-soliton solution, $N = 2$, $\lambda_3 = -\lambda_1$, $\lambda_4 = -\lambda_2$.

$$\lambda_1 = \rho_1\,\mathrm{e}^{\mathrm{i}\beta_1} = \mu_1 + \mathrm{i}\nu_1, \qquad \lambda_2 = \rho_2\,\mathrm{e}^{\mathrm{i}\beta_2} = \mu_2 + \mathrm{i}\nu_2, \tag{72}$$

$$\begin{aligned} c_{10} &= \frac{b_{10}}{\dot{a}(\lambda_1)} = b_{10}\frac{\zeta_1^2 - \overline{\zeta_1^2}}{2\zeta_1}\cdot\frac{\zeta_1^2 - \overline{\zeta_2^2}}{\zeta_1^2 - \zeta_2^2}\cdot\frac{\zeta_1^2}{\overline{\zeta_1^2}}\cdot\frac{\zeta_2^2}{\overline{\zeta_2^2}} \\ c_{20} &= \frac{b_{20}}{\dot{a}(\lambda_2)} = b_{20}\frac{\zeta_2^2 - \overline{\zeta_2}}{2\zeta_2}\cdot\frac{\zeta_2^2 - \overline{\zeta_1^2}}{\zeta_2^2 - \zeta_1^2}\cdot\frac{\zeta_1^2}{\overline{\zeta_1^2}}\cdot\frac{\zeta_2^2}{\overline{\zeta_2^2}} \end{aligned} \tag{73}$$

$$\begin{aligned} &f_j^2 = 2c_{j0}\,\mathrm{e}^{\mathrm{i}2\lambda_j^2 x}\,\mathrm{e}^{\mathrm{i}4\lambda_j^4 t} \qquad j = 1, 2 \quad (\text{cf } (62)) \\ &b_{j0}\,\mathrm{e}^{\mathrm{i}2\lambda_j^2 x + i4\lambda_j^4 t} \equiv \mathrm{e}^{-\theta_j}\,\mathrm{e}^{\mathrm{i}\varphi_j}, \qquad j = 1, 2 \end{aligned} \tag{74}$$

where

$$\begin{aligned} \theta_j &= 4\mu_j\nu_j\big[x - x_{j0} + 4\big(\mu_j^2 - \nu_j^2\big)t\big] \\ \varphi_j &= 2\big(\mu_j^2 - \nu_j^2\big)x + \big[4\big(\mu_j^2 - \nu_j^2\big)^2 - 16\mu_j^2\nu_j^2\big]\cdot t + \alpha_{j0} \end{aligned} \tag{75}$$

and b_{j0} is absorbed into the soliton center and the initial phase by

$$b_{j0} = \mathrm{e}^{4\mu_j\nu_j x_{j0}}\,\mathrm{e}^{\mathrm{i}\alpha_{j0}}, \qquad j = 1, 2. \tag{76}$$

and we get

$$
\begin{aligned}
A_2 &= \sum_{\substack{n_1=1,2\\ m_1=0}} \Omega_1(n_1,0)\Omega_2(0,n_1) + \sum_{\substack{n_1=1,n_2=2\\ m_1=0,m_2=1,2}} \Omega_1(n_1,n_2;0,m_2)\Omega_2(0,m_2;n_1,n_2) \\
&= +\Omega_1(1;0)\Omega_2(0;1) + \Omega_1(2;0)\Omega_2(0;2) \\
&\quad + \Omega_1(1,2;0,1)\Omega_2(0,1;1,2) + \Omega_1(1,2;0,2)\Omega_2(0,2;1,2) \\
&= f_1^2 + f_2^2 - |f_1|^4 f_2^2 \frac{\left(\lambda_1^2-\lambda_2^2\right)^2\overline{\lambda_1^2}}{\left(\lambda_1^2-\overline{\lambda_1^2}\right)^2\left(\overline{\lambda_1^2}-\lambda_2^2\right)^2} - |f_2|^4 f_1^2 \frac{\left(\lambda_1^2-\lambda_2^2\right)^2\overline{\lambda_2^2}}{\left(\overline{\lambda_2^2}-\lambda_1^2\right)^2\left(\lambda_2^2-\overline{\lambda_2^2}\right)^2} \\
&= \lambda_1(1-\mathrm{e}^{-\mathrm{i}4\beta_1})\frac{\lambda_1^2-\overline{\lambda_2^2}}{\lambda_1^2-\lambda_2^2}\,\mathrm{e}^{\mathrm{i}4(\beta_1+\beta_2)}\,\mathrm{e}^{-\theta_1}\,\mathrm{e}^{\mathrm{i}\varphi_1} + \lambda_2(1-\mathrm{e}^{-\mathrm{i}4\beta_2})\frac{\overline{\lambda_1^2}-\lambda_2^2}{\lambda_1^2-\lambda_2^2}\,\mathrm{e}^{\mathrm{i}4(\beta_1+\beta_2)}\,\mathrm{e}^{-\theta_2}\,\mathrm{e}^{\mathrm{i}\varphi_2} \\
&\quad + \left[\lambda_1(1-\mathrm{e}^{-\mathrm{i}4\beta_1})\,\mathrm{e}^{-\mathrm{i}2\beta_2}\frac{\overline{\lambda_1^2}-\lambda_2^2}{\overline{\lambda_1^2}-\overline{\lambda_2^2}}\,\mathrm{e}^{-\theta_2-\mathrm{i}\varphi_2} + \lambda_2(1-\mathrm{e}^{-\mathrm{i}4\beta_2})\,\mathrm{e}^{-\mathrm{i}2\beta_1}\frac{\lambda_1^2-\overline{\lambda_2^2}}{\overline{\lambda_1^2}-\overline{\lambda_2^2}}\,\mathrm{e}^{-\theta_1-\mathrm{i}\varphi_1}\right] \\
&\quad \bullet\ \mathrm{e}^{-(\theta_1+\theta_2)}\,\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)}\,\mathrm{e}^{\mathrm{i}4(\beta_1+\beta_2)} \\
&= \mathrm{i}2\left|\frac{\lambda_1^2-\overline{\lambda_2^2}}{\lambda_1^2-\lambda_2^2}\right| \bullet [\rho_1\sin 2\beta_1\,\mathrm{e}^{\mathrm{i}(\phi-\alpha)}\,\mathrm{e}^{\mathrm{i}(3\beta_1+4\beta_2)}\,\mathrm{e}^{-\theta_1+\mathrm{i}\varphi_1} + \rho_2\sin 2\beta_2\,\mathrm{e}^{-\mathrm{i}(\phi+\alpha)}\,\mathrm{e}^{\mathrm{i}(4\beta_1+3\beta_2)}\,\mathrm{e}^{-\theta_2+\mathrm{i}\varphi_2} \\
&\quad + \rho_1\sin 2\beta_1\,\mathrm{e}^{-\mathrm{i}(\phi-\alpha)}\,\mathrm{e}^{\mathrm{i}(3\beta_1+2\beta_2)}\,\mathrm{e}^{-2\theta_2-\theta_1}\cdot\mathrm{e}^{\mathrm{i}\varphi_1} + \rho_2\sin 2\beta_2\,\mathrm{e}^{\mathrm{i}(\phi+\alpha)}\,\mathrm{e}^{\mathrm{i}(2\beta_1+3\beta_2)}\,\mathrm{e}^{-2\theta_1-\theta_2}\cdot\mathrm{e}^{\mathrm{i}\varphi_2}]
\end{aligned} \tag{77}
$$

where

$$
\begin{aligned}
\phi &= \arg\left(\lambda_1^2-\overline{\lambda_2^2}\right) = \arctan\left(\frac{\rho_1^2\sin 2\beta_1+\rho_2^2\sin 2\beta_2}{\rho_1^2\cos 2\beta_1-\rho_2^2\cos 2\beta_2}\right) \\
\alpha &= \arg\left(\lambda_1^2-\lambda_2^2\right) = \arctan\left(\frac{\rho_1^2\sin 2\beta_1-\rho_2^2\sin 2\beta_2}{\rho_1^2\cos 2\beta_1-\rho_2^2\cos 2\beta_2}\right)
\end{aligned} \tag{78}
$$

and

$$
\begin{aligned}
\bar{D}_2 &= 1 + \sum_{r=1}^{2}\sum_{1\leqslant n_1<n_2\leqslant 2}\sum_{1\leqslant m_1<m_2\leqslant 2} Q_1(n_1,\ldots,n_r;m_1,\ldots,m_r)\,Q_2(m_1,\ldots,m_r;n_1,\ldots,n_r) \\
&= 1 + Q_1(n_1=1;m_1=1)\,Q_2(m_1=1,n_1=1) + Q_1(n_1=1;m_1=2)\,Q_2(m_1=2,n_1=1) \\
&\quad + Q_1(n_1=2;m_1=1)\,Q_2(m_1=1,n_1=2) + Q_1(n_1=2;m_1=2)\,Q_2(m_1=2,n_1=2) \\
&\quad + Q_1(n_1=1,n_2=2;m_1=1,m_2=2)\,Q_2(m_1=1,m_2=2;n_1=1,n_2=2) \\
&= 1 - |f_1|^4\frac{\lambda_1^2}{\left(\lambda_1^2-\overline{\lambda_1^2}\right)^2} - |f_2|^4\frac{\lambda_2^2}{\left(\lambda_2^2-\overline{\lambda_2^2}\right)^2} - f_1^2\overline{f_2^2}\frac{\lambda_1^2}{\left(\lambda_1^2-\overline{\lambda_2^2}\right)^2} - \overline{f_2^2}f_2^2\frac{\lambda_2^2}{\left(\overline{\lambda_1^2}-\lambda_2^2\right)^2} \\
&\quad + |f_1f_2|^4\cdot\frac{\lambda_1^2\lambda_2^2\left(\overline{\lambda_1^2}-\overline{\lambda_2^2}\right)^2\left(\lambda_1^2-\lambda_2^2\right)^2}{\left(\lambda_1^2-\overline{\lambda_1^2}\right)^2\left(\lambda_1^2-\overline{\lambda_2^2}\right)^2\left(\lambda_2^2-\overline{\lambda_1^2}\right)^2\left(\lambda_2^2-\overline{\lambda_2^2}\right)^2}
\end{aligned} \tag{79}
$$

$$
\begin{aligned}
&= 1 + \left|\frac{\lambda_1^2-\overline{\lambda_2^2}}{\lambda_1^2-\lambda_2^2}\right|^2(\mathrm{e}^{-\mathrm{i}2\beta_1}\,\mathrm{e}^{-2\theta_1} + \mathrm{e}^{-\mathrm{i}2\beta_2}\mathrm{e}^{-2\theta_2}) \\
&\quad + \left(1-\left|\frac{\lambda_1^2-\overline{\lambda_2^2}}{\lambda_1^2-\lambda_2^2}\right|^2\right)\mathrm{e}^{-(\theta_1+\theta_2)}\mathrm{e}^{-\mathrm{i}(\beta_1+\beta_2)}\left[\frac{\rho_1}{\rho_2}\mathrm{e}^{\mathrm{i}(\varphi_2-\varphi_1)} + \frac{\rho_2}{\rho_1}\mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)}\right] + \mathrm{e}^{-\mathrm{i}2(\beta_1+\beta_2)}\,\mathrm{e}^{-2(\theta_1+\theta_2)}
\end{aligned} \tag{80}
$$

where

$$\left|\frac{\lambda_1^2-\overline{\lambda_2^2}}{\lambda_1^2-\lambda_2^2}\right|^2=\frac{(\rho_1/\rho_2-\rho_2/\rho_1)^2+4\sin^2(\beta_1+\beta_2)}{(\rho_1/\rho_2-\rho_2/\rho_1)^2+4\sin^2(\beta_1-\beta_2)}. \tag{80a}$$

Substituting (77) and (80) into formula (49), we thus get the two-soliton solution to the DNLS equation with VBC

$$\bar{u}_2=-\mathrm{i}2\frac{A_2D_2}{\overline{D_2^2}}. \tag{81}$$

Once again we find that, up to a permitted global constant phase factor, the above two-soliton solution is equivalent to that obtained in [27], verifying the validity of our formula of N-soliton solution and the reliability of those linear algebra techniques. As a matter of fact, a general and strict demonstration of our revised IST for DNLS equation with VBC has been given in one paper [28] by use of the Liouville theorem.

6. The asymptotic behaviors of the N-soliton solution

The complex conjugate of expression (49) gives the explicit expression of the N-soliton solution as

$$u_N=\mathrm{i}2\frac{\bar{A}_N\bar{D}_N}{D_N^2}. \tag{82}$$

Without loss of generality, for $\lambda_n=\mu_n+\mathrm{i}\upsilon_n$, $V_n=-4(\mu_n^2-\upsilon_n^2)$, $n=1,2,\ldots,N$, we assume $V_1<V_2<\cdots<V_n<\cdots V_N$, and define the nth vicinity area as

$$\Lambda_n: x-x_{n_0}-V_nt\sim 0, \qquad (n=1,2,\ldots,N)$$

As $t\to-\infty$, N vicinity areas Λ_n, $n=1,2\ldots,N$, queue up in a descending series

$$\Lambda_N,\Lambda_{N-1},\cdots,\Lambda_1, \tag{83}$$

and in the vicinity of Λ_n, we have (note that $\kappa_j>0$)

$$\theta_j=4\kappa_j(x-x_{j0}-V_jt) \quad\to\quad \begin{cases}+\infty, & \text{for } j>n\\ -\infty, & \text{for } j<n.\end{cases} \tag{84}$$

Here the complex constant $2c_n$ in expression (62) has been absorbed into $\mathrm{e}^{-\theta_n}\,\mathrm{e}^{\mathrm{i}\varphi_n}$ by redefinition of the soliton center x_{n0} and the initial phase α_{n0}.

Introducing a typical factor $F_n=-\frac{\mathrm{e}^{-2\theta_n}}{\left(\lambda_n^2-\overline{\lambda_n^2}\right)^2}>0$, $n=1,2,\ldots,N$; then

$$D_n(1,2,\ldots,n)=\prod_{j=1}^{n}\overline{\lambda_j^2}F_j\prod_{l<m}\left|\frac{\lambda_l^2-\lambda_m^2}{\lambda_l^2-\overline{\lambda}_m^2}\right|^4 \qquad \text{where}\quad l,m\in\{1,2,\ldots,n\} \tag{85}$$

$$\begin{aligned}D&\simeq D_{n-1}(1,2,\ldots,n-1)+D_n(1,2,\ldots,n)\\ &=\left(1+\overline{\lambda_n^2}F_n\prod_{j=1}^{n-1}\left|\frac{\lambda_j^2-\lambda_n^2}{\lambda_j^2-\overline{\lambda}_n^2}\right|^4\right)D_{n-1}(1,2,\ldots,n-1)\end{aligned} \tag{86}$$

and

$$\begin{aligned}\overline{A}&\simeq\overline{A}_n(1,2,\ldots,n;0,1,2,\ldots,n-1)\\ &=D_{n-1}\,\mathrm{e}^{-\theta_n}\,\mathrm{e}^{-\mathrm{i}\varphi_n}\prod_{j=1}^{n-1}\frac{\left(\overline{\lambda_j^2}-\overline{\lambda_n^2}\right)^2}{\left(\lambda_j^2-\overline{\lambda_n^2}\right)^2}\,\mathrm{e}^{\mathrm{i}4\beta_j}.\end{aligned} \tag{87}$$

In the vicinity of Λ_n,

$$u = \mathrm{i}2\frac{\overline{AD}}{D^2} \simeq u_1\big(\theta_n + \Delta\theta_n^{(-)}, \varphi_n + \Delta\varphi_n^{(-)}\big). \tag{88}$$

Here

$$\Delta\theta_n^{(-)} = 2\sum_{j=1}^{n-1} \ln\left|\frac{\lambda_j^2 - \overline{\lambda_n^2}}{\lambda_j^2 - \lambda_n^2}\right|^2 \tag{89}$$

$$\begin{aligned}\Delta\varphi_n^{(-)} &= -\sum_{j=1}^{n-1}\left\{\arg\left[\frac{(\overline{\lambda_j^2} - \overline{\lambda_n^2})^2}{\left(\lambda_j^2 - \overline{\lambda_n^2}\right)^2}\right] + 4\beta_j\right\} \\ &= 2\sum_{j=1}^{n-1}\left[\arg\left(\lambda_j^2 - \overline{\lambda_n^2}\right) - \arg\left(\overline{\lambda_j^2} - \overline{\lambda_n^2}\right) - 2\beta_j\right]\end{aligned} \tag{90}$$

then

$$u_N \simeq \sum_{n=1}^{N} u_1\big(\theta_n + \Delta\theta_n^{(-)}, \varphi_n + \Delta\varphi_n^{(-)}\big). \tag{91}$$

Each $u_1(\theta_n, \varphi_n)$, $(1, 2, \ldots, n)$ is a one-soliton solution characterized by one parameter λ_n, moving to the positive direction along the x-axis, queuing up in a series with descending order number n as in series (83).

As $t \to \infty$, in the vicinity of Λ_n we have (note that $\kappa_j > 0$)

$$\theta_j = 4\kappa_j(x - x_{j0} - V_j t) \to \begin{cases} -\infty, & \text{for} \quad j > n \\ +\infty, & \text{for} \quad j < n \end{cases} \tag{92}$$

$$\begin{aligned} D &\simeq D_{N-n}(n+1, n+2, \ldots, N) + D_{N-n+1}(n, n+1, \ldots, N) \\ &= \left(1 + \overline{\lambda_n^2} F_n \prod_{j=n+1}^{N}\left|\frac{\lambda_j^2 - \lambda_n^2}{\lambda_j^2 - \overline{\lambda}_n^2}\right|^4\right) D_{N-n}(n+1, n+2, \ldots, N)\end{aligned} \tag{93}$$

$$\begin{aligned}\overline{A} &\simeq \overline{A_{N-n+1}}(n, n+1, \ldots, N; 0, n+1, n+2, \ldots, N) \\ &= D_{N-n}(n+1, n+2, \ldots, N)\mathrm{e}^{-\theta_n}\,\mathrm{e}^{-\mathrm{i}\varphi_n}\prod_{j=n+1}^{N}\frac{\left(\overline{\lambda_j^2} - \overline{\lambda_n^2}\right)^2}{\left(\lambda_j^2 - \overline{\lambda_n^2}\right)^2}\frac{\lambda_j^2}{\overline{\lambda}_j^2}.\end{aligned} \tag{94}$$

So as $t \to \infty$, in the vicinity of Λ_n,

$$u = \mathrm{i}2\frac{\overline{AD}}{D^2} \simeq u_1\left(\theta_n + \Delta\theta_n^{(+)}, \varphi_n + \Delta\varphi_n^{(+)}\right) \tag{95}$$

$$\Delta\theta_n^{(+)} = 2\sum_{j=n+1}^{N} \ln\left|\frac{\lambda_j^2 - \overline{\lambda_n^2}}{\lambda_j^2 - \lambda_n^2}\right|^2 \tag{96}$$

$$\begin{aligned}\Delta\varphi_n^{(+)} &= -\sum_{j=n+1}^{N}\left\{\arg\left[\frac{\left(\overline{\lambda_j^2} - \overline{\lambda_n^2}\right)^2}{\left(\lambda_j^2 - \overline{\lambda_n^2}\right)^2}\right] + 4\beta_j\right\} \\ &= 2\sum_{j=n+1}^{N}\left[\arg\left(\lambda_j^2 - \overline{\lambda_n^2}\right) - \arg\left(\overline{\lambda_j^2} - \overline{\lambda_n^2}\right) - 2\beta_j\right].\end{aligned} \tag{97}$$

Then as $t \to \infty$,

$$u_N \simeq \sum_{n=1}^{N} u_1 \left(\theta_n + \Delta\theta_n^{(+)}, \varphi_n + \Delta\varphi_n^{(+)}\right). \tag{98}$$

That is to say, the N-soliton solution can be viewed as N well-separated exact one-solitons, queuing up in a series with ascending order number n:

$$\Lambda_1, \Lambda_2, \ldots, \Lambda_N. \tag{99}$$

In the course going from $t \to -\infty$ to $t \to \infty$, the nth one-soliton overtakes the solitons from the first to $n-1$th, and is overtaken by the solitons from $n+1$th to Nth. In the meantime, due to collisions, the nth soliton had a total forward shift $\Delta\theta_n^{(-)}/\kappa_n$ from exceeding those slower soliton from the first to $n-1$th, and got a total backward shift$\Delta\theta_n^{(+)}/\kappa_n$ from being exceeded by those faster solitons from $n+1$th to Nth, and just equaled the summation of shifts due to each collision between two solitons, together with a total phase shift$\Delta\varphi_n$, that is,

$$\Delta x_n = \left|\Delta\theta_n^{(+)} - \Delta\theta_n^{(-)}\right| / \kappa_n \tag{100}$$

$$\Delta\varphi_n = \Delta\varphi_n^{(+)} - \Delta\varphi_n^{(-)}. \tag{101}$$

7. The N-soliton solution to the MNLS equation

A nonlinear Schrödinger equation including the nonlinear dispersion term expressed as

$$\mathrm{i}\partial_t \upsilon + \partial_{xx}\upsilon + \mathrm{i}\alpha\partial_x(|\upsilon|^2\upsilon) + 2\beta|\upsilon|^2\upsilon = 0 \tag{102}$$

is also integrable [29], and called the modified nonlinear Schrödinger (MNLS) equation. It is well known that the MNLS equation well describes transmission of femtosecond pulses in optical fibers [9–13] and is related to the DNLS equation by a gauge-like transformation [29] formulated as

$$\upsilon(x,t) = u(X,T)\,\mathrm{e}^{\mathrm{i}2\rho X + \mathrm{i}4\rho^2 T} \tag{103}$$

with

$$x = \alpha^{-1}(X + 4\rho T), \qquad t = \alpha^{-2}T, \qquad \rho = \beta\alpha^{-2}. \tag{104}$$

Using a method that is analogous to [29], and applying above gauge-like transformation to equations (102), the MNLS equation with VBC can be transformed into the DNLS equation with VBC,

$$\mathrm{i}\partial_T u + \partial_{XX} u + \mathrm{i}\partial_X(|u|^2 u) = 0 \tag{105}$$

with $u = u(X,T)$.

So according to (103), the N-soliton solution to the MNLS equation can also be attained by a gauge-like transformation from that of the DNLS equation.

8. Concluding remarks

Based upon improved inverse scattering transform recently proposed in our group and the Zakharov–Shabat equation for the DNLS equation with VBC considered anew, the N-soliton solution to the DNLS equation with VBC has been derived by means of standard IST and some special linear algebra techniques. The one- and two-soliton solutions have been cited as two typical examples in illustration of the general formula of the N-soliton solution. It is found to be perfectly in agreement with that obtained in other papers of ours [26, 30] based on a pure Marchenko formalism without the need of IST. The demonstration of the revised IST considered anew for the DNLS equation with VBC has also been given by use of the Liouville theorem [28].

Meanwhile, the asymptotic behaviors of the N-soliton solution have been discussed in detail. In the limit of $t \to \pm\infty$, the N-soliton solution can be viewed as the summation of N single solitons with a definite displacement and phase shift of each soliton in the whole course of elastic collisions.

An interesting conclusion is drawn that, besides a permitted well-known constant global phase factor, there is also an undetermined constant complex parameter b_{n0} before each of the typical soliton factor $e^{-\theta_n} e^{i\varphi_n}$ $(n = 1, 2, \ldots, N)$. It can be absorbed into $e^{-\theta_n} e^{i\varphi_n}$ by a redefinition of soliton center and its initial phase factor. This kind of arbitrariness is in correspondence with the unfixed initial conditions of the DNLS equation. Finally, we indicate that the exact N-soliton solution to the DNLS equation can be converted to that of the MNLS equation by gauge-like transformation.

The newly revised IST technique for the DNLS equation with VBC provides substantial foundation for its direct perturbation theory, because the Jost functions given by the new IST are of regular properties and normal asymptotic behaviors.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under grant nos. 10375041 and 10705022.

Appendix

Some useful formulae of linear algebra are as follows.

If A_1 and A_2 are $N \times 1$ matrices, A is a regular $N \times N$ matrix, then

$$A_1^T A^{-1} A_2 = \frac{\det\left(A + A_2 A_1^T\right)}{\det(A)} - 1. \tag{A.1}$$

For a squared matrix B

$$\det(I + B) = 1 + \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant N} B(n_1, n_2, \ldots, n_r) \tag{A.2}$$

where $B(n_1, n_2, \ldots, n_r)$ is the rth-order principal minor of B.

For an $N \times N$ matrix Q_1 and an $N \times N$ matrix Q_2,

$$\begin{aligned}\det(I + Q_1 Q_2) &= 1 + \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant N} \Omega_r(n_1, n_2, \ldots, n_r) \\ &= 1 + \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < \cdots < n_r \leqslant N} \sum_{1 \leqslant m_1 < \cdots < m_r \leqslant N} Q_1(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) \\ &\quad \times Q_2(m_1, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r),\end{aligned} \tag{A.3}$$

where $Q_1(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)$ denotes a minor, which is the determinant of a submatrix of Q_1 consisting of elements belonging to not only rows $(n_1, n_2, \ldots, n_r)$ but also columns $(m_1, m_2, \ldots, m_r)$.

The above formula also holds for the case of $\det(I + \Omega_1 \Omega_2)$ with Ω_1 to be an $N \times (N + 1)$ matrix and Ω_2 an $(N + 1) \times N$ matrix.

For a squared matrix C with elements $C_{jk-} = f_j g_k (x_j - y_k)^{-1}$,

$$\det(C) = \prod_j f_j g_j \prod_{j<j',k<k'} (x_j - x_{j'})(y_{k'} - y_k) \prod_{j,k} (x_j - y_k)^{-1}. \qquad \text{(A.4)}$$

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